BLOCK TYPE SYMMETRY OF BIGRADED TODA HIERARCHY

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Abstract. In this paper, we define Orlov-Schulman's operators M_L, M_R, and then use them to construct the additional symmetries of the bigraded Toda hierarchy (BTH). We further show that these additional symmetries form an interesting infinite dimensional Lie algebra known as a Block type Lie algebra, whose structure theory and representation theory have recently received much attention in literature. By acting on two different spaces under the weak W-constraints we find in particular two representations of this Block type Lie algebra.

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Contents

1. Introduction

The Toda lattice equation as a completely integrable system was introduced by Toda [1] to describe an infinite system of masses on a line that interact through an exponential force.

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Inspired by the Sato theory on the Kadomtsev-Petviashvili (KP) hierarchy [2, 3], the two dimensional Toda hierarchy was constructed by Ueno and Takasaki [4] with the help of difference operators and infinite dimensional Lie algebras. The one dimensional Toda hierarchy (TH) was also studied under the reduction condition $L + L^{-1} = M + M^{-1}$ [4] or $L = M$ on two Lax operators L and M. The bigraded Toda hierarchy (BTH) of (N, M)-type(or simply the (N, M)-BTH) is the generalized Toda hierarchy whose infinite Lax matrix has N upper and M lower nonzero diagonals[5]. After continuous interpolation, N and M correspond to the highest and lowest powers of Laurent polynomials which is the Lax operator. The (N, M)-BTH can be naturally considered as a reduction of the two-dimensional Toda hierarchy by imposing an algebraic relation to those two Lax operators(see [4, 5]). One dimensional Toda hierarchy(i.e. $(1, 1)$-BTH) and the BTH have been shown to be related to many mathematical and physical fields such as the inverse scattering method, finite and infinite dimensional algebras, classical and quantum field theories and so on. Recently, in [6, 7], the interpolated Toda lattice hierarchy was generalized to the so-called extended Toda hierarchy (ETH) for considering its application on the topological field theory. In [8], the TH and ETH were further generalized to the extended bigraded Toda hierarchy (EBTH) by considering $N+M$ dependent variables $\{u_{N-1}, u_{N-2}, \cdots, u_1, u_0, u_{-1,}, \cdots, u_{-M}\}$ in the Lax operator $\mathcal{L}$. This new model has been expected [8] that it might be relevant for applications in describing the Gromov-Witten invariants. In fact the dispersionless case of that model has been proposed in [9] because the dispersionless EBTH can be obtained from the dispersionless KP hierarchy. In [10], the Hirota bilinear equations (HBEs) of EBTH have been given conjecturally and proved that it governs the Gromov-Witten theory of orbifold c_{km}. In [11], the authors generalize the Sato theory to the EBTH and give the Hirota bilinear equations in terms of vertex operators whose coefficients take values in the algebra of differential operators. In [12], a geometric structure associated with Frobenius manifold of 2D Toda hierarchy was introduced. Furthermore, motivated by the potential applications of the BTH, which is also defined by omitting the extended logarithmic flows of the EBTH, in the theory of the matrix models, it is necessary and interesting to explore its algebraic structure from the point of view of the additional symmetry.

Additional symmetries of KP hierarchy were given by Orlov and Shulman [13] through two novel operators Γ and M, which can be used to form a centerless W algebra. Based on this work, there exist many extensive results (e.g., [14–22]) on the additional symmetries of the KP hierarchy, Toda hierarchy, 2-D Toda hierarchy, BKP hierarchy and CKP hierarchy. Particularly, the representations of the infinite dimensional Virasoro algebra and W algebra have been derived by using the actions on the Lax operator, the wave function and the τ function of additional symmetry flows. These results inspire us to search new infinite dimensional algebras from the additional symmetry flows of the BTH. So the purpose of this paper is to give the additional symmetries of the BTH and then identify its algebraic structure. In [23], the additional symmetries of KP hierarchy were generalized to a $W_{1+\infty}$ algebra. However, the commutative relations of the $W_{1+\infty}$ algebra are rather complicated. The algebra under consideration in this paper is very simple and elegant, which is an infinite dimensional Lie algebra $\mathcal{B}$, known as a *Block type Lie algebra*, introduced by Block [24] around 50 years ago.

This kind of Lie algebra is an interesting object in the structure theory and representation theory of Lie algebras, partly due to its close relation with the Virasoro algebra and the Virasoro-like algebra (e.g., [25–31]). To our best knowledge, this is the first time to bring Block type Lie algebras to the integrable system. In particular, we obtain two representations of the Lie algebra $\mathcal{B}$ on two spaces of functions P_L, P_R respectively (Theorem 6.1), which (to the best of our knowledge) are the first known examples of representations of $\mathcal{B}$ with the actions of generating elements of $\mathcal{B}$ being explicitly given. Another quite different feature of additional symmetries in the BTH is that $M \triangleq M_L - M_R$ operator for constructing additional flows commutes with Lax operator $\mathcal{L}$, i.e., $[\mathcal{L}, M] = 0$. This is a crucial fact to find Block type Lie algebra here. Note that $[L, M] = 1$ holds for other known integrable systems such as KP hierarchy.

The paper is organized as follows. In Section 2, the two dimensional Toda hierarchy and its reductions are introduced explicitly by which we can define the BTH later. In Section 3, the definition of the BTH and its Sato equation are introduced. In Section 4, we define Orlov-Schulman's M_L, M_R operators and prove their linear equations. The additional symmetries and related equations of the BTH will be given in Section 5, meanwhile we prove that the additional symmetries have a nice structure of a Block type Lie algebra. In Section 6, we give some specific actions of that Block type additional flows on spaces of functions P_L, P_R which further lead to representations of this Block type Lie algebra under so-called weak W-constraints. Section 7 is devoted to conclusions and discussions.

2. The two-dimensional Toda hierarchy and its reductions

In this section, we will show that the BTH is just a general reduction of the two-dimensional Toda hierarchy whose special reduction leads to original Toda hierarchy.

Firstly we will introduce the definition of the two-dimensional Toda hierarchy in interpolated form as following.

The two-dimensional Toda hierarchy [4] can be defined by the following two Lax operators,

$$L = \Lambda + a_0 + a_{-1}\Lambda^{-1} + a_{-2}\Lambda^{-2} + \dots, \tag{2.1}$$

$$\bar{L} = \bar{a}_{-1}\Lambda^{-1} + \bar{a}_0 + \bar{a}_1\Lambda^1 + \bar{a}_2\Lambda^2 + \dots, \tag{2.2}$$

where Λ represents the shift operator with $\Lambda := e^{\epsilon\partial_x}$ and "ϵ" is called the string coupling constant, i.e. for any function $f(x)$

$$\Lambda f(x) = f(x + \epsilon).$$

The coefficients a_n and $\bar{a}_k$ are the functions of x and $\{(x_n, y_n) : n = 1, 2, \dots\}$. Then the Lax representation of the two-dimensional Toda hierarchy is given by the set of infinite number of equations for $n = 1, 2, \dots$,

$$\frac{\partial L}{\partial x_n} = [L^n_+, L], \qquad \frac{\partial L}{\partial y_n} = [\bar{L}^n_-, L], \tag{2.3}$$

$$\frac{\partial \bar{L}}{\partial x_n} = [L_+^n, \bar{L}], \qquad \frac{\partial \bar{L}}{\partial y_n} = [\bar{L}_-^n, \bar{L}], \tag{2.4}$$

where L_+^n represents the part of L^n with non-negative powers in Λ, and $\bar{L}_-^n$ represents the part of $\bar{L}^n$ with negative powers in Λ. In particular, the Lax equations for $n = 1$ provide the system for $(a_0, \bar{a}_{-1})$,

$$\begin{cases} \dfrac{\partial \bar{a}_{-1}(x)}{\partial x_1} & = \bar{a}_{-1}(x)(a_0(x) - a_0(x-\epsilon)), \\ \dfrac{\partial a_0(x)}{\partial y_1} & = \bar{a}_{-1}(x) - \bar{a}_{-1}(x+\epsilon). \end{cases} \tag{2.5}$$

With the function $u(x)$ defined by $a_0(x) = \partial_{x_1} u(x)$ and $\bar{a}_{-1}(x) = e^{u(x)-u(x-\epsilon)}$, the two-dimensional Toda equation,

$$\frac{\partial^2 u(x)}{\partial x_1 \partial y_1} = e^{u(x)-u(x-\epsilon)} - e^{u(x+\epsilon)-u(x)} \tag{2.6}$$

is given.

The 1-D Toda equation is given by the reduction,

$$L = \bar{L} = \Lambda + a_0 + a_{-1}\Lambda^{-1}. \tag{2.7}$$

Then the Lax equations for x_1 and y_1 give

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right) L = [L_+ + \bar{L}_-, L] = [L, L] = 0. \tag{2.8}$$

This implies that L does not depend on the variable $s_1 := x_1 + y_1$. Then the two-dimensional Toda equation is reduced to the 1-D Toda equation, and with $t_1 := x_1 - y_1, \bar{u}(x) = -u(x)$, we have the standard 1-D Toda equation as

$$\frac{\partial^2 \bar{u}(x)}{\partial t_1^2} = e^{\bar{u}(x-\epsilon)-\bar{u}(x)} - e^{\bar{u}(x)-\bar{u}(x+\epsilon)}. \tag{2.9}$$

In the next subsection, we generalize the reduction (2.7) to the general Laurent polynomial, i.e.

$$L^N = \bar{L}^M, \quad N, M \in \mathcal{N}, \tag{2.10}$$

which defines the (N, M)-BTH(L and $\bar{L}$ will correspond to fractional powers of Lax operator of the BTH (see (3.14))).

3. Bigraded Toda hierarchy

The Lax form of the BTH(i.e. (N, M)-BTH) can be introduced as [8]. For that we need to introduce firstly the Lax operator

$$\mathcal{L} = \Lambda^N + u_{N-1}\Lambda^{N-1} + \cdots + u_{-M}\Lambda^{-M} \tag{3.1}$$

(where $N, M \geq 1$ are two fixed positive integers). The variables u_j are functions of the real variable x. The Lax operator $\mathcal{L}$ can be written in two different ways by dressing the shift operator

$$\mathcal{L} = \mathcal{P}_L \Lambda^N \mathcal{P}_L^{-1} = \mathcal{P}_R \Lambda^{-M} \mathcal{P}_R^{-1}. \tag{3.2}$$

Equation (3.2) is quite important because it gives the reduction condition (2.10) of the BTH from the two-dimensional Toda hierarchy.

The two dressing operators have the following form

$$\mathcal{P}_L = 1 + w_1 \Lambda^{-1} + w_2 \Lambda^{-2} + \dots, \tag{3.3}$$

$$\mathcal{P}_R = \tilde{w}_0 + \tilde{w}_1 \Lambda + \tilde{w}_2 \Lambda^2 + \dots, \tag{3.4}$$

and their inverses have form

$$\mathcal{P}_L^{-1} = 1 + \Lambda^{-1} w_1' + \Lambda^{-2} w_2' + \dots, \tag{3.5}$$

$$\mathcal{P}_R^{-1} = \tilde{w_0'} + \Lambda \tilde{w_1'} + \Lambda^2 \tilde{w_2'} + \dots. \tag{3.6}$$

The coefficients $\{w_i, \tilde{w}_i, w_i', \tilde{w_i'}, i \geq 0\}$ will be used more later in calculating the representation of Block algebra. The pair is unique up to multiplying $\mathcal{P}_L$ and $\mathcal{P}_R$ from the right by operators in the form $1 + a_1\Lambda^{-1} + a_2\Lambda^{-2} + \dots$ and $\tilde{a}_0 + \tilde{a}_1\Lambda + \tilde{a}_2\Lambda^2 + \dots$ respectively with coefficients independent of x. From the first identity of (3.2), the relations of $u_i, -M \leq i \leq N-1$ and $w_j, j \geq 1$ are as follows (see [11])

$$\begin{aligned}
u_{N-1} &= w_1(x) - w_1(x+N\epsilon), && (3.7)\\
u_{N-2} &= w_2(x) - w_2(x+N\epsilon) - (w_1(x) - w_1(x+N\epsilon))w_1(x+(N-1)\epsilon), && (3.8)\\
u_{N-3} &= w_3(x) - w_3(x+N\epsilon) \\
&\quad -[w_2(x) - w_2(x+N\epsilon) - (w_1(x) - w_1(x+N\epsilon))w_1(x+(N-1)\epsilon)]w_1(x+(N-2)\epsilon) \\
&\quad -(w_1(x) - w_1(x+N\epsilon))w_2(x+(N-1)\epsilon), && (3.9)\\
&\cdots\cdots\cdots
\end{aligned}$$

Moreover, by using the second identity of (3.2), we can also easily get the relations of u_i and $\tilde{w}_j$ formally as follows

$$\begin{aligned}
u_{-M} &= \frac{\tilde{w}_0(x)}{\tilde{w}_0(x-M\epsilon)}, && (3.10)\\
u_{-M+1} &= \frac{\tilde{w}_1(x) - \frac{\tilde{w}_0(x)}{\tilde{w}_0(x-M\epsilon)}\tilde{w}_1(x-M\epsilon)}{\tilde{w}_0(x-(M-1)\epsilon)}, && (3.11)\\
u_{-M+2} &= \frac{\tilde{w}_2(x) - \frac{\tilde{w}_0(x)}{\tilde{w}_0(x-M\epsilon)}\tilde{w}_2(x-M\epsilon) - \frac{\tilde{w}_1(x) - \frac{\tilde{w}_0(x)}{\tilde{w}_0(x-M\epsilon)}\tilde{w}_1(x-M\epsilon)}{\tilde{w}_0(x-(M-1)\epsilon)}\tilde{w}_1(x-(M-1)\epsilon)}{\tilde{w}_0(x-(M-2)\epsilon)}, && (3.12)\\
&\cdots\cdots\cdots
\end{aligned}$$

These relations above will be used in the calculation later. Given any difference operator $A = \sum_k A_k\Lambda^k$, the positive and negative projections are given by $A_+ = \sum_{k\geq 0} A_k\Lambda^k$ and $A_- = \sum_{k<0} A_k\Lambda^k$.

To write out explicitly the Lax equations of the BTH, fractional powers $\mathcal{L}^{\frac{1}{N}}$ and $\mathcal{L}^{\frac{1}{M}}$ are defined by

$$\mathcal{L}^{\frac{1}{N}} = \Lambda + \sum_{k\leq 0} a_k\Lambda^k, \qquad \mathcal{L}^{\frac{1}{M}} = \sum_{k\geq -1} b_k\Lambda^k,$$

with the relations

$$(\mathcal{L}^{\frac{1}{N}})^N = (\mathcal{L}^{\frac{1}{M}})^M = \mathcal{L}. \tag{3.13}$$

Acting on free function $\lambda^{\frac{x}{\epsilon}}$, these two fraction powers can be seen as two different locally expansions around zero and infinity respectively. It was stressed that $\mathcal{L}^{\frac{1}{N}}$ and $\mathcal{L}^{\frac{1}{M}}$ are two different operators even if $N = M(N, M \geq 2)$ in [8] due to two different dressing operators. They can also be expressed as following

$$\mathcal{L}^{\frac{1}{N}} = \mathcal{P}_L\Lambda\mathcal{P}_L^{-1}, \qquad \mathcal{L}^{\frac{1}{M}} = \mathcal{P}_R\Lambda^{-1}\mathcal{P}_R^{-1}. \tag{3.14}$$

Similar to [8], the BTH can be defined as following.

Definition 3.1. The bigraded Toda hierarchy consists of a system of flows given in the Lax pair formalism by

$$\partial_{t_{\gamma,n}}\mathcal{L} = [A_{\gamma,n}, \mathcal{L}] \tag{3.15}$$

for $\gamma = N, N-1, N-2, \dots, -M+1$ and $n \geq 0$. The operators $A_{\gamma,n}$ are defined by

$$A_{\gamma,n} = (\mathcal{L}^{n+1-\frac{\gamma-1}{N}})_+ \quad \text{for} \quad \gamma = N, N-1, \dots, 2, 1, \tag{3.16a}$$

$$A_{\gamma,n} = -(\mathcal{L}^{n+1+\frac{\gamma}{M}})_- \quad \text{for} \quad \gamma = 0, -1, \dots, -M+1, \tag{3.16b}$$

and $\partial_{t_{\gamma,n}}$ is defined as $\frac{\partial}{\partial t_{\gamma,n}}$.

The only difference from [8] is that we cancel the extended flows and add the flow when $\gamma = 1$. The flow when $\gamma = 1$ is in fact the Toda hierarchy which is also the flow when $\gamma = 0$.

Particularly for $N = 1 = M$, this hierarchy coincides with the one dimensional Toda hierarchy. When $N = 1, M = 2$, the BTH leads the following primary equations

$$\partial_{1,0}\mathcal{L} = [\Lambda + u_0, \mathcal{L}], \tag{3.17}$$

and

$$\partial_{-1,0}\mathcal{L} = -[e^{(1+\Lambda^{-1})^{-1}\log u_{-2}}\Lambda^{-1}, \mathcal{L}], \tag{3.18}$$

which further lead to

$$\begin{cases} \partial_{1,0}u_0(x) = u_{-1}(x+\epsilon) - u_{-1}(x), \\ \partial_{1,0}u_{-1}(x) = u_{-2}(x+\epsilon) - u_{-2}(x) + u_{-1}(x)(u_0(x) - u_0(x-\epsilon)), \\ \partial_{1,0}u_{-2}(x) = u_{-2}(x)(u_0(x) - u_0(x-2\epsilon)), \end{cases} \tag{3.19}$$

and

$$\begin{cases} \partial_{-1,0}u_0(x) = e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x+\epsilon)} - e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x)}, \\ \partial_{-1,0}u_{-1}(x) = e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x)}(u_0(x) - u_0(x-\epsilon)), \\ \partial_{-1,0}u_{-2}(x) = u_{-1}(x)e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x-\epsilon)} - e^{(1+\Lambda^{-1})^{-1}\log u_{-2}(x)}u_{-1}(x-\epsilon). \end{cases} \tag{3.20}$$

Obviously equation (3.20) contains infinite multiplication because of nonlocal term $(1+\Lambda^{-1})^{-1}\log u_{-2}(x)$ which comes from the fractional power of the Lax operator.

Set $N = 2$ and $M = 1$, the equations (3.15) are as follows

$$\partial_{2,0}\mathcal{L} = [\Lambda + (1+\Lambda)^{-1}u_1(x), \mathcal{L}], \tag{3.21}$$

$$\partial_{1,0}\mathcal{L} = [\Lambda^2 + u_1\Lambda + u_0, \mathcal{L}], \tag{3.22}$$

which further lead to the following concrete equations

$$\begin{cases} \partial_{2,0}u_1(x) = u_1(x+\epsilon) - u_1(x) + u_1(x)(1-\Lambda)(1+\Lambda)^{-1}u_1(x), \\ \partial_{2,0}u_0(x) = u_{-1}(x+\epsilon) - u_{-1}(x), \\ \partial_{2,0}u_{-1}(x) = u_{-1}(x)(1-\Lambda^{-1})(1+\Lambda)^{-1}u_1(x), \end{cases} \tag{3.23}$$

$$\begin{cases} \partial_{1,0}u_1(x) = u_{-1}(x+2\epsilon) - u_{-1}(x), \\ \partial_{1,0}u_0(x) = u_{-2}(x+2\epsilon) - u_{-2}(x) + u_1(x)u_{-1}(x+\epsilon) - u_{-1}u_1(x-\epsilon), \\ \partial_{1,0}u_{-1}(x) = u_{-1}(x)(u_0(x) - u_0(x-\epsilon)). \end{cases} \tag{3.24}$$

Notice that the nonlocal term $(1+\Lambda)^{-1}u_1(x)$ also comes from the fractional power of Lax operator in equations above. Therefore appearance of nonlocal term is an important property of the BTH. We can also get more equations when N and M take other integer values but we shall not mention them here because our central consideration in this paper is the Block type additional symmetries of the BTH.

For the convenience to derive the Sato equations, the following operators will be defined as in [8, 11]:

$$B_{\gamma,n} := \begin{cases} \mathcal{L}^{n+1-\frac{\gamma-1}{N}} & \text{for } \gamma = N \dots 1, \\ \mathcal{L}^{n+1+\frac{\gamma}{M}} & \text{for } \gamma = 0 \dots - M + 1. \end{cases} \tag{3.25}$$

Before introducing the Sato equation, the following proposition [8] need to be given firstly.

Proposition 3.2. *The following two identities hold*

$$\partial_{t_{\gamma,n}}\mathcal{L}^{\frac{1}{N}} = [-(B_{\gamma,n})_-, \mathcal{L}^{\frac{1}{N}}], \tag{3.26}$$

$$\partial_{t_{\gamma,n}}\mathcal{L}^{\frac{1}{M}} = [(B_{\gamma,n})_+, \mathcal{L}^{\frac{1}{M}}]. \tag{3.27}$$

Proof. See [8, 11]. □

Using the proposition above, one can obtain the following proposition, lemma and theorem, which are results of [8, 11].

Proposition 3.3. *If L satisfies the Lax equation* (3.15)*, then we have the following Zakharov-Shabat equation*

$$\partial_{t_{\beta,n}}(A_{\alpha,m}) - \partial_{t_{\alpha,m}}(A_{\beta,n}) + [A_{\alpha,m}, A_{\beta,n}] = 0, \tag{3.28}$$

for $-M+1 \leq \alpha, \beta \leq N$, $m, n \geq 0$.

Using the Zakharov-Shabat equation (3.28) one can obtain the following lemma.

Lemma 3.4. ([11]) *The following Zakharov-Shabat equations hold*

$$\partial_{\beta,n}(B_{\alpha,m})_- - \partial_{\alpha,m}(B_{\beta,n})_- - [(B_{\alpha,m})_-, (B_{\beta,n})_-] = 0, \tag{3.29}$$

$$-\partial_{\beta,n}(B_{\alpha,m})_+ + \partial_{\alpha,m}(B_{\beta,n})_+ - [(B_{\alpha,m})_+, (B_{\beta,n})_+] = 0, \tag{3.30}$$

where, $-M+1 \leq \alpha, \beta \leq N$, $m, n \geq 0$.

Using Lemma 3.4 and the Lax equation, one can then obtain the following theorem.

Theorem 3.5. ([11]) *L is a solution to the BTH if and only if there is a pair of dressing operators $\mathcal{P}_L$ and $\mathcal{P}_R$, which satisfy the following Sato equations:*

$$\partial_{\gamma,n}\mathcal{P}_L = -(B_{\gamma,n})_-\mathcal{P}_L, \tag{3.31}$$

$$\partial_{\gamma,n}\mathcal{P}_R = (B_{\gamma,n})_+\mathcal{P}_R, \tag{3.32}$$

where, $-M+1 \leq \gamma \leq N$, $n \geq 0$.

The dressing operators satisfying Sato equations (3.31) and (3.32) will be called wave operators later.

After the preparation above, it is time to introduce Orlov-Schulman's operators which is included in the next section.

4. Orlov-Schulman's M_L, M_R operators

In order to give the additional symmetries of the BTH, we define the Orlov-Schulman's M_L, M_R operators by

$$M_L = \mathcal{P}_L\Gamma_L\mathcal{P}_L^{-1}, \qquad M_R = \mathcal{P}_R\Gamma_R\mathcal{P}_R^{-1}, \tag{4.33}$$

where

$$\Gamma_L = \frac{x}{N\epsilon}\Lambda^{-N} + \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})\Lambda^{N(n-\frac{\alpha-1}{N})}t_{\alpha,n}, \tag{4.34}$$

$$\Gamma_R = -\frac{x}{M\epsilon}\Lambda^{M} - \sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})\Lambda^{-M(n+\frac{\beta}{M})}t_{\beta,n}. \tag{4.35}$$

A direct calculation shows that the operators M_L and M_R satisfy the following theorem.

Proposition 4.1. *Operators $\mathcal{L}$,M_L and M_R satisfy the following identities*

$$[\mathcal{L}, M_L] = 1, \qquad [\mathcal{L}, M_R] = 1, \tag{4.36}$$

$$\partial_{t_{\gamma,n}} M_L = [A_{\gamma,n}, M_L], \qquad \partial_{t_{\gamma,n}} M_R = [A_{\gamma,n}, M_R], \tag{4.37}$$

$$\frac{\partial M_L^n \mathcal{L}^k}{\partial t_{\gamma,n}} = [A_{\gamma,n}, M_L^n \mathcal{L}^k], \qquad \frac{\partial M_R^n \mathcal{L}^k}{\partial t_{\gamma,n}} = [A_{\gamma,n}, M_R^n \mathcal{L}^k], \tag{4.38}$$

where, $-M+1 \le \gamma \le N, n \ge 0$.

Proof. Firstly we prove (4.36) by dressing the following two identities using $\mathcal{P}_L$ and $\mathcal{P}_R$ separately

$$\begin{aligned}
[\Lambda^N, \Gamma_L] &= [\Lambda^N, \frac{x}{N\epsilon}\Lambda^{-N} + \sum_{n\ge 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})\Lambda^{N(n-\frac{\alpha-1}{N})}t_{\alpha,n}] \\
&= 1, \\
[\Lambda^{-M}, \Gamma_R] &= [\Lambda^{-M}, -\frac{x}{M\epsilon}\Lambda^{M} - \sum_{n\ge 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})\Lambda^{-M(n+\frac{\beta}{M})}t_{\beta,n}] \\
&= 1.
\end{aligned}$$

For the proof of (4.37), we need to prove

$$\partial_{t_{\alpha,n}} M_L = [(B_{\alpha,n})_+, M_L], \tag{4.39}$$

$$\partial_{t_{\alpha,n}} M_R = [(B_{\alpha,n})_+, M_R], \tag{4.40}$$

$$\partial_{t_{\beta,n}} M_L = [-(B_{\beta,n})_-, M_L], \tag{4.41}$$

$$\partial_{t_{\beta,n}} M_R = [-(B_{\beta,n})_-, M_R], \tag{4.42}$$

where, $1 \le \alpha \le N, -M+1 \le \beta \le 0$.

Let us consider the following bracket

$$[\partial_{t_{\alpha,n}} - \Lambda^{N(n+1-\frac{\alpha-1}{N})}, \Gamma_L] = 0, \tag{4.43}$$

which can be easily got by a direct computation. By dressing both sides of the identity above using the operator $\mathcal{P}_L$, the left part of (4.43) becomes

$$[\mathcal{P}_L \partial_{t_{\alpha,n}} \mathcal{P}_L^{-1} - \mathcal{L}^{n+1-\frac{\alpha-1}{N}}, \mathcal{P}_L \Gamma_L \mathcal{P}_L^{-1}] = [\partial_{t_{\alpha,n}} - (B_{\alpha,n})_+, M_L],$$

which leads to (4.39).

For the proof of (4.41), we consider the following bracket

$$[\partial_{t_{\beta,n}}, \Gamma_L] = 0. \tag{4.44}$$

By dressing the identity above in the same way we can get

$$[\mathcal{P}_L \partial_{t_{\beta,n}} \mathcal{P}_L^{-1}, \mathcal{P}_L \Gamma_L \mathcal{P}_L^{-1}] = [\partial_{t_{\beta,n}} + (B_{\beta,n})_-, M_L],$$

which leads to

$$\partial_{t_{\beta,n}} M_L = [-(B_{\beta,n})_-, M_L].$$

Similarly, we can prove (4.40) and (4.42) by dressing the identity

$$[\partial_{t_{\alpha,n}}, \Gamma_R] = 0,$$

and

$$[\partial_{t_{\beta,n}} + \Lambda^{-M(n+1+\frac{\beta}{M})}, \Gamma_R] = 0,$$

respectively, through the dressing operator $\mathcal{P}_R$. Using proved (4.37) and Lax equation (3.15), we can prove (4.38) easily. □

The equations in Proposition 4.1 can be realized by linear equations in the following proposition. To simplify the theorem, we first introduce two functions $w_L(t,\lambda)$ and $w_R(t,\lambda)$ which have forms

$$w_L(t,\lambda) = \mathcal{P}_L(x,\Lambda)e^{\xi_L(t,\lambda)} = P_L(x,\Lambda)e^{\xi_L(t,\lambda)}, \tag{4.45}$$

$$w_R(t,\lambda) = \mathcal{P}_R(x,\Lambda)e^{\xi_R(t,\lambda)} = P_R(x,\Lambda)e^{\xi_R(t,\lambda)}, \tag{4.46}$$

where

$$\xi_L(t,\lambda) = \sum_{n\geq 0}\sum_{\alpha=1}^{N} \lambda^{(n+1-\frac{\alpha-1}{N})} t_{\alpha,n} + \frac{x}{N\epsilon}\log\lambda, \tag{4.47}$$

$$\xi_R(t,\lambda) = -\sum_{n\geq 0}\sum_{\beta=-M+1}^{0} \lambda^{-(n+1+\frac{\beta}{M})} t_{\beta,n} + \frac{x}{M\epsilon}\log\lambda. \tag{4.48}$$

Functions $w_L(t,\lambda)$ and $w_R(t,\lambda)$ will be called wave functions. P_L and P_R are called symbols of $\mathcal{P}_L$ and $\mathcal{P}_R$ respectively.

Proposition 4.2. *The wave functions $w_L(t,\lambda)$ and $w_R(t,\lambda)$ satisfy the following linear equations*

$$\begin{cases} \mathcal{L}w_L(t,\lambda) = & \lambda w_L(t,\lambda), \\ M_L w_L(t,\lambda) = & \partial_\lambda w_L(t,\lambda), \\ \partial_{t_{\gamma,n}} w_L(t,\lambda) = & A_{\gamma,n} w_L(t,\lambda), \end{cases} \tag{4.49}$$

$$\begin{cases} \mathcal{L}w_R(t,\lambda) = & \lambda^{-1} w_R(t,\lambda), \\ M_R w_R(t,\lambda) = & \partial_{\lambda^{-1}} w_R(t,\lambda), \\ \partial_{t_{\gamma,n}} w_R(t,\lambda) = & A_{\gamma,n} w_R(t,\lambda), \end{cases} \tag{4.50}$$

where, $-M+1 \leq \gamma \leq N, n \geq 0$.

Proof. Because

$$\Lambda^N \exp(\frac{x}{N\epsilon}\log\lambda) = \lambda\exp(\frac{x}{N\epsilon}\log\lambda),$$

so

$$\Lambda^N e^{\xi_L(t,\lambda)} = \lambda e^{\xi_L(t,\lambda)}.$$

Then using the definition of dressing operators, we can get

$$\begin{aligned}\mathcal{L}w_L(t,\lambda) &= \mathcal{L}\mathcal{P}_L(x,\Lambda)e^{\xi_L(t,\lambda)}\\ &= \mathcal{P}_L(x,\Lambda)\Lambda^N e^{\xi_L(t,\lambda)}\\ &= \lambda w_L(t,\lambda).\end{aligned}$$

Similarly we can do the following computation

$$\begin{aligned}M_L w_L(t,\lambda) &= M_L\mathcal{P}_L(x,\Lambda)e^{\xi_L(t,\lambda)}\\ &= \mathcal{P}_L(x,\Lambda)(\frac{x}{N\epsilon}\Lambda^{-N} + \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})\Lambda^{N(n-\frac{\alpha-1}{N})}t_{\alpha,n})e^{\xi_L(t,\lambda)}\\ &= \mathcal{P}_L(x,\Lambda)(\frac{x}{N\epsilon}\lambda^{-1} + \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})\lambda^{n-\frac{\alpha-1}{N}}t_{\alpha,n})e^{\xi_L(t,\lambda)}\\ &= \mathcal{P}_L(x,\Lambda)\frac{\partial}{\partial\lambda}e^{\xi_L(t,\lambda)}\\ &= \frac{\partial}{\partial\lambda}w_L(t,\lambda).\end{aligned}$$

For the time flows of linear functions, we have to consider flows when $1 \leq \gamma \leq N$ and flows when $-M+1 \leq \gamma \leq 0$ separately. Setting $1 \leq \alpha \leq N, -M+1 \leq \beta \leq 0$, and considering the relations

$$e^{\xi_L(t,\lambda)} = \exp(\sum_{n\geq 0}\sum_{\alpha=1}^{N}\Lambda^{N(n+1-\frac{\alpha-1}{N})}t_{\alpha,n})e^{\frac{x}{N\epsilon}\log\lambda},$$

we do the derivative as follows

$$\begin{aligned}&\partial_{\alpha,n}w_L(t,\lambda)\\ =\ &(\partial_{\alpha,n}\mathcal{P}_L(x,\Lambda))\exp(\sum_{n\geq 0}\sum_{\alpha=1}^{N}\Lambda^{N(n+1-\frac{\alpha-1}{N})}t_{\alpha,n})e^{\frac{x}{N\epsilon}\log\lambda}\\ &+\mathcal{P}_L(x,\Lambda)\Lambda^{N(n+1-\frac{\alpha-1}{N})}\exp(\sum_{n\geq 0}\sum_{\alpha=1}^{N}\Lambda^{N(n+1-\frac{\alpha-1}{N})}t_{\alpha,n})e^{\frac{x}{N\epsilon}\log\lambda}\\ =\ &-(B_{\alpha,n})_-\mathcal{P}_L(x,\Lambda)e^{\xi_L(t,\lambda)} + \mathcal{L}^{n+1-\frac{\alpha-1}{N}}\mathcal{P}_L(x,\Lambda)e^{\xi_L(t,\lambda)}\\ =\ &(B_{\alpha,n})_+w_L(t,\lambda).\end{aligned}$$

Because $\xi_L(t,\lambda)$ does not depend the time variables $t_{\beta,n}$, we have

$$\begin{aligned}\partial_{\beta,n}w_L(t,\lambda) &= (\partial_{\beta,n}\mathcal{P}_L(x,\Lambda))\exp(\sum_{n\geq 0}\sum_{\alpha=1}^{N}\Lambda^{N(n+1-\frac{\alpha-1}{N})}t_{\alpha,n})e^{\frac{x}{N\epsilon}\log\lambda}\\ &= -(B_{\beta,n})_-\mathcal{P}_L(x,\Lambda)e^{\xi_L(t,\lambda)}\\ &= -(B_{\beta,n})_-w_L(t,\lambda).\end{aligned}$$

Now the proof of (4.49) is finished. In the similar way, using

$$\Lambda^{-M}\exp(\frac{x}{M\epsilon}\log\lambda) = \lambda^{-1}\exp(\frac{x}{M\epsilon}\log\lambda),$$

and

$$-\frac{x}{M\epsilon}\lambda\exp(\frac{x}{M\epsilon}\log\lambda) = \partial_{\lambda^{-1}}\exp(\frac{x}{M\epsilon}\log\lambda),$$

we can prove the equations in (4.50). □

Moreover, on the space of wave functions $w_L(t,\lambda)$ and $w_R(t,\lambda)$, identities $[\mathcal{L},M_L]=1$, $[\mathcal{L},M_R]=1$ and $[\lambda,\partial_\lambda]=-1$ induce an anti-isomorphism between $(\mathcal{L},M_L)$, $(\mathcal{L},M_R)$ and $(\lambda,\partial_\lambda)$, i.e.,

$$M_L^m\mathcal{L}^l w_L(t,\lambda) = \lambda^l\left(\partial_\lambda^m w_L(t,\lambda)\right), \tag{4.51}$$

$$\mathcal{L}^l M_L^m w_L(t,\lambda) = \partial_\lambda^m\left(\lambda^l w_L(t,\lambda)\right), m,l\in Z_+. \tag{4.52}$$

$$M_R^m\mathcal{L}^l w_R(t,\lambda) = \lambda^{-l}\left(\partial_{\lambda^{-1}}^m w_R(t,\lambda)\right), \tag{4.53}$$

$$\mathcal{L}^l M_R^m w_R(t,\lambda) = \partial_{\lambda^{-1}}^m\left(\lambda^{-l} w_R(t,\lambda)\right), m,l\in Z_+. \tag{4.54}$$

5. Additional symmetries of BTH

We are now to define the additional flows, and then to prove that they are symmetries, which are called additional symmetries of the BTH. We introduce additional independent variables $t^*_{m,l}$ and define the actions of the additional flows on the wave operators as

$$\frac{\partial\mathcal{P}_L}{\partial t^*_{m,l}} = -\left((M_L-M_R)^m\mathcal{L}^l\right)_-\mathcal{P}_L,\quad \frac{\partial\mathcal{P}_R}{\partial t^*_{m,l}} = \left((M_L-M_R)^m\mathcal{L}^l\right)_+\mathcal{P}_R, \tag{5.55}$$

where $m\geq 0, l\geq 0$. The following theorem shows that the definition (5.55) is compatible with reduction condition (3.2) of the BTH.

Theorem 5.1. *The additional flows* (5.55) *preserve reduction condition* (3.2).

Proof. By performing the derivative on $\mathcal{L}$ dressed by $\mathcal{P}_L$ and using the additional flow about $\mathcal{P}_L$ in (5.55), we get

$$\begin{aligned}(\partial_{t^*_{m,l}}\mathcal{L}) &= (\partial_{t^*_{m,l}}\mathcal{P}_L)\ \Lambda^N\mathcal{P}_L^{-1} + \mathcal{P}_L\ \Lambda^N\ (\partial_{t_{m,l}}\mathcal{P}_L^{-1})\\ &= -((M_L-M_R)^m\mathcal{L}^l)_-\mathcal{P}_L\ \Lambda^N\ \mathcal{P}_L^{-1} - \mathcal{P}_L\ \Lambda^N\mathcal{P}_L^{-1}\ (\partial_{t^*_{m,l}}\mathcal{P}_L)\ \mathcal{P}_L^{-1}\end{aligned}$$

$$\begin{aligned} &= -((M_L - M_R)^m \mathcal{L}^l)_- \mathcal{L} + \mathcal{L}((M_L - M_R)^m \mathcal{L}^l)_- \\ &= -[((M_L - M_R)^m \mathcal{L}^l)_-, \mathcal{L}]. \end{aligned}$$

Similarly, we perform the derivative on $\mathcal{L}$ dressed by $\mathcal{P}_R$ and use the additional flow about $\mathcal{P}_R$ in (5.55) to get the following

$$\begin{aligned} (\partial_{t^*_{m,l}} \mathcal{L}) &= (\partial_{t^*_{m,l}} \mathcal{P}_R)\ \Lambda^{-M} \mathcal{P}_R^{-1} + \mathcal{P}_R\ \Lambda^{-M}\ (\partial_{t_{m,l}} \mathcal{P}_R^{-1}) \\ &= ((M_L - M_R)^m \mathcal{L}^l)_+ \mathcal{P}_R\ \Lambda^{-M}\ \mathcal{P}_R^{-1} - \mathcal{P}_R\ \Lambda^{-M} \mathcal{P}_R^{-1}\ (\partial_{t^*_{m,l}} \mathcal{P}_R)\ \mathcal{P}_R^{-1} \\ &= ((M_L - M_R)^m \mathcal{L}^l)_+ \mathcal{L} - \mathcal{L}((M_L - M_R)^m \mathcal{L}^l)_+ \\ &= [((M_L - M_R)^m \mathcal{L}^l)_+, \mathcal{L}]. \end{aligned}$$

Because

$$[M_L - M_R, \mathcal{L}] = 0, \tag{5.56}$$

therefore

$$\frac{\partial \mathcal{L}}{\partial t^*_{m,l}} = [-\left((M_L - M_R)^m \mathcal{L}^l\right)_-, \mathcal{L}] = [\left((M_L - M_R)^m \mathcal{L}^l\right)_+, \mathcal{L}], \tag{5.57}$$

which gives the compatibility of additional flow of BTH with reduction condition (3.2). □

Similarly, we can take derivatives on dressing structure of M_L and M_R to get the following proposition.

Proposition 5.2. *The additional derivatives act on M_L, M_R as*

$$\frac{\partial M_L}{\partial t^*_{m,l}} = [-\left((M_L - M_R)^m \mathcal{L}^l\right)_-, M_L], \tag{5.58}$$

$$\frac{\partial M_R}{\partial t^*_{m,l}} = [\left((M_L - M_R)^m \mathcal{L}^l\right)_+, M_R]. \tag{5.59}$$

Proof. By performing the derivative on M_L given in (4.33), there exists a similar derivative as $\partial_{t^*_{m,l}} \mathcal{L}$, i.e.,

$$\begin{aligned} (\partial_{t^*_{m,l}} M_L) &= (\partial_{t^*_{m,l}} \mathcal{P}_L)\ \Gamma_L \mathcal{P}_L^{-1} + \mathcal{P}_L\ \Gamma_L\ (\partial_{t^*_{m,l}} \mathcal{P}_L^{-1}) \\ &= -((M_L - M_R)^m \mathcal{L}^l)_- \mathcal{P}_L\ \Gamma_L\ \mathcal{P}_L^{-1} - \mathcal{P}_L\ \Gamma_L \mathcal{P}_L^{-1}\ (\partial_{t^*_{m,l}} \mathcal{P}_L)\ \mathcal{P}_L^{-1} \\ &= -((M_L - M_R)^m \mathcal{L}^l)_- M_L + M_L((M_L - M_R)^m \mathcal{L}^l)_- \\ &= -[((M_L - M_R)^m \mathcal{L}^l)_-, M_L]. \end{aligned}$$

Here the fact that Γ_L does not depend on the additional variables $t^*_{m,l}$ has been used. Other identities can also be obtained in a similar way. □

By two propositions above, the following corollary can be easily got.

Corollary 5.3. *For $1 \le \alpha \le N, -M+1 \le \beta \le 0, \quad n, m, l \ge 0$, the following identities hold*

$$\frac{\partial \mathcal{L}^{\frac{n}{N}}}{\partial t^*_{m,l}} = [-((M_L - M_R)^m \mathcal{L}^l)_-, \mathcal{L}^{\frac{n}{N}}], \qquad \frac{\partial B_{\alpha,n}}{\partial t^*_{m,l}} = [-((M_L - M_R)^m \mathcal{L}^l)_-, B_{\alpha,n}], \tag{5.60}$$

$$\frac{\partial \mathcal{L}^{\frac{n}{M}}}{\partial t^*_{m,l}} = [((M_L - M_R)^m \mathcal{L}^l)_+, \mathcal{L}^{\frac{n}{M}}], \qquad \frac{\partial B_{\beta,n}}{\partial t^*_{m,l}} = [((M_L - M_R)^m \mathcal{L}^l)_+, B_{\beta,n}], \tag{5.61}$$

$$\frac{\partial M_L^n}{\partial t^*_{m,l}} = [-((M_L - M_R)^m \mathcal{L}^l)_-, M_L^n], \qquad \frac{\partial M_R^n}{\partial t^*_{m,l}} = [((M_L - M_R)^m \mathcal{L}^l)_+, M_R^n], \tag{5.62}$$

$$\frac{\partial M_L^n \mathcal{L}^k}{\partial t^*_{m,l}} = -[((M_L - M_R)^m \mathcal{L}^l)_-, M_L^n \mathcal{L}^k], \quad \frac{\partial M_R^n \mathcal{L}^k}{\partial t^*_{m,l}} = [((M_L - M_R)^m \mathcal{L}^l)_+, M_R^n \mathcal{L}^k]. \tag{5.63}$$

Proof. First we present the proof of the first equation. Considering the dressing relation

$$\mathcal{L}^{\frac{n}{N}} = \mathcal{P}_L \Lambda^n \mathcal{P}_L^{-1},$$

and (5.55), we can get relations

$$\frac{\partial \mathcal{L}^{\frac{n}{N}}}{\partial t^*_{m,l}} = [-((M_L - M_R)^m \mathcal{L}^l)_-, \mathcal{L}^{\frac{n}{N}}],$$

which further leads to the second identity in (5.60). Similarly, by relations

$$\mathcal{L}^{\frac{n}{M}} = \mathcal{P}_R \Lambda^{-n} \mathcal{P}_R^{-1}, \qquad M_L = \mathcal{P}_L \Gamma_L \mathcal{P}_L^{-1}, \qquad M_R = \mathcal{P}_R \Gamma_R \mathcal{P}_R^{-1},$$

and (5.55), we can get other relations in similar ways. $\square$

With Proposition 5.2 and Corollary 5.3, the following theorem can be proved.

Theorem 5.4. *The additional flows $\partial_{t^*_{m,l}}$ commute with the bigraded Toda hierarchy flows $\partial_{t_{\gamma,n}}$, i.e.,*

$$[\partial_{t^*_{m,l}}, \partial_{t_{\gamma,n}}]\Phi = 0, \tag{5.64}$$

where Φ can be $\mathcal{P}_L$, $\mathcal{P}_R$ or L, $-M+1 \le \gamma \le N$ and $\partial_{t^*_{m,l}} = \frac{\partial}{\partial t^*_{m,l}}, \partial_{t_{\gamma,n}} = \frac{\partial}{\partial t_{\gamma,n}}$.

Proof. According to the definition,

$$[\partial_{t^*_{m,l}}, \partial_{t_{\gamma,n}}]\mathcal{P}_L = \partial_{t^*_{m,l}}(\partial_{t_{\gamma,n}} \mathcal{P}_L) - \partial_{t_{\gamma,n}}(\partial_{t^*_{m,l}} \mathcal{P}_L),$$

and using the actions of the additional flows and the bigraded Toda flows on $\mathcal{P}_L$, for $1 \le \alpha \le N$, we have

$$\begin{aligned}
[\partial_{t^*_{m,l}}, \partial_{t_{\alpha,n}}]\mathcal{P}_L &= -\partial_{t^*_{m,l}}((B_{\alpha,n})_- \mathcal{P}_L) + \partial_{t_{\alpha,n}}\left(((M_L - M_R)^m \mathcal{L}^l)_-^m \mathcal{P}_L\right) \\
&= -(\partial_{t^*_{m,l}} B_{\alpha,n})_- \mathcal{P}_L - (B_{\alpha,n})_- (\partial_{t^*_{m,l}} \mathcal{P}_L) \\
&\quad + [\partial_{t_{\alpha,n}}((M_L - M_R)^m \mathcal{L}^l)]_- \mathcal{P}_L + ((M_L - M_R)^m \mathcal{L}^l)_- (\partial_{t_{\alpha,n}} \mathcal{P}_L).
\end{aligned}$$

Using (5.55) and Theorem 4.1, it equals

$$[\partial_{t^*_{m,l}}, \partial_{t_{\alpha,n}}]\mathcal{P}_L = [\left((M_L - M_R)^m \mathcal{L}^l\right)_-, B_{\alpha,n}]_- \mathcal{P}_L + (B_{\alpha,n})_- \left((M_L - M_R)^m \mathcal{L}^l\right)_- \mathcal{P}_L$$

$$
\begin{aligned}
&+[(B_{\alpha,n})_+,(M_L-M_R)^m\mathcal{L}^l]_-\mathcal{P}_L-((M_L-M_R)^m\mathcal{L}^l)_-(B_{\alpha,n})_-\mathcal{P}_L\\
=\;&[((M_L-M_R)^m\mathcal{L}^l)_-,B_{\alpha,n}]_-\mathcal{P}_L-[(M_L-M_R)^m\mathcal{L}^l,(B_{\alpha,n})_+]_-\mathcal{P}_L\\
&+[(B_{\alpha,n})_-,((M_L-M_R)^m\mathcal{L}^l)_-]\mathcal{P}_L\\
=\;&0.
\end{aligned}
$$

Similarly, using (5.55) and Theorem 4.1, we can prove the additional flows commute with flows $\partial_{t_{\beta,n}}$ with $-M+1\le\beta\le 0$ in the sense of acting on $\mathcal{P}_L$. Of course, using (5.55), (3.31), (3.32) and Theorem 4.1, we can also prove the additional flows commute with all flows of BTH in the sense of acting on $\mathcal{P}_R, L$. Here we also give the proof for commutativity of additional symmetries with $\partial_{t_{\beta,n}}$, where $-M+1\le\beta\le 0$. To be a little different from the proof above, we let the Lie bracket act on $\mathcal{P}_R$,

$$
\begin{aligned}
[\partial_{t^*_{m,l}},\partial_{t_{\beta,n}}]\mathcal{P}_R &= \partial_{t^*_{m,l}}((B_{\beta,n})_+\mathcal{P}_R)-\partial_{t_{\beta,n}}\big(((M_L-M_R)^m\mathcal{L}^l)_+\mathcal{P}_R\big)\\
&= (\partial_{t^*_{m,l}}B_{\beta,n})_+\mathcal{P}_R+(B_{\beta,n})_+(\partial_{t^*_{m,l}}\mathcal{P}_R)\\
&\quad -(\partial_{t_{\beta,n}}((M_L-M_R)^m\mathcal{L}^l))_+\mathcal{P}_R-((M_L-M_R)^m\mathcal{L}^l)_+(\partial_{t_{\beta,n}}\mathcal{P}_R),
\end{aligned}
$$

which further leads to

$$
\begin{aligned}
[\partial_{t^*_{m,l}},\partial_{t_{\beta,n}}]\mathcal{P}_R &= [((M_L-M_R)^m\mathcal{L}^l)_+,B_{\beta,n}]_+\mathcal{P}_R+(B_{\beta,n})_+((M_L-M_R)^m\mathcal{L}^l)_+\mathcal{P}_R\\
&\quad +[(B_{\beta,n})_-,(M_L-M_R)^m\mathcal{L}^l]_+\mathcal{P}_R-((M_L-M_R)^m\mathcal{L}^l)_+(B_{\beta,n})_+\mathcal{P}_R\\
&= [((M_L-M_R)^m\mathcal{L}^l)_+,B_{\beta,n}]_+\mathcal{P}_R+[(B_{\beta,n})_-,(M_L-M_R)^m\mathcal{L}^l]_+\mathcal{P}_R\\
&\quad +[(B_{\beta,n})_+,((M_L-M_R)^m\mathcal{L}^l)_+]\mathcal{P}_R\\
&= [((M_L-M_R)^m\mathcal{L}^l)_+,B_{\beta,n}]_+\mathcal{P}_R+[B_{\beta,n},((M_L-M_R)^m\mathcal{L}^l)_+]_+\mathcal{P}_R=0.
\end{aligned}
$$

In the proof above, $[(B_{\gamma,n})_+,((M_L-M_R)^m\mathcal{L}^l)]_-=[(B_{\gamma,n})_+,((M_L-M_R)^m\mathcal{L}^l)_-]_-$ has been used, since $[(B_{\gamma,n})_+,((M_L-M_R)^m\mathcal{L}^l)_+]_-=0$. ☐

The commutative property in Theorem 5.4 means that additional flows are symmetries of the BTH. Since they are symmetries, it is natural to consider the algebraic structures among these additional symmetries. So we obtain the following important theorem.

Theorem 5.5. *The additional flows $\partial_{t^*_{m,l}}(m>0,l\ge 0)$ form a Block type Lie algebra with the following relation*

$$
[\partial_{t^*_{m,l}},\partial_{t^*_{n,k}}]=(km-nl)\partial^*_{m+n-1,k+l-1}, \tag{5.65}
$$

which holds in the sense of acting on $\mathcal{P}_L$, $\mathcal{P}_R$ or $\mathcal{L}$ and $m,n\ge 0$, $l,k\ge 0$.

Proof. By using (5.55), we get

$$
\begin{aligned}
[\partial_{t^*_{m,l}},\partial_{t^*_{n,k}}]\mathcal{P}_L &= \partial_{t^*_{m,l}}(\partial_{t^*_{n,k}}\mathcal{P}_L)-\partial_{t^*_{n,k}}(\partial_{t^*_{m,l}}\mathcal{P}_L)\\
&= -\partial_{t^*_{m,l}}\big(((M_L-M_R)^n\mathcal{L}^k)_-\mathcal{P}_L\big)+\partial_{t^*_{n,k}}\big(((M_L-M_R)^m\mathcal{L}^l)_-\mathcal{P}_L\big)\\
&= -(\partial_{t^*_{m,l}}(M_L-M_R)^n\mathcal{L}^k)_-\mathcal{P}_L-((M_L-M_R)^n\mathcal{L}^k)_-(\partial_{t^*_{m,l}}\mathcal{P}_L)
\end{aligned}
$$

$$+(\partial_{t^*_{n,k}}(M_L-M_R)^m\mathcal{L}^l)_-\mathcal{P}_L+((M_L-M_R)^m\mathcal{L}^l)_-(\partial_{t^*_{n,k}}\mathcal{P}_L).$$

On the account of Proposition5.2, we further get

$$\begin{aligned}
&[\partial_{t^*_{m,l}},\partial_{t^*_{n,k}}]\mathcal{P}_L\\
=\ & -\Big[\sum_{p=0}^{n-1}(M_L-M_R)^p(\partial_{t^*_{m,l}}(M_L-M_R))(M_L-M_R)^{n-p-1}\mathcal{L}^k+(M_L-M_R)^n(\partial_{t^*_{m,l}}\mathcal{L}^k)\Big]_-\mathcal{P}_L\\
&-((M_L-M_R)^n\mathcal{L}^k)_-(\partial_{t^*_{m,l}}\mathcal{P}_L)\\
&+\Big[\sum_{p=0}^{m-1}(M_L-M_R)^p(\partial_{t^*_{n,k}}(M_L-M_R))(M_L-M_R)^{m-p-1}\mathcal{L}^l+(M_L-M_R)^m(\partial_{t^*_{n,k}}\mathcal{L}^l)\Big]_-\mathcal{P}_L\\
&+((M_L-M_R)^m\mathcal{L}^l)_-(\partial_{t^*_{n,k}}\mathcal{P}_L)\\
=\ & [(nl-km)(M_L-M_R)^{m+n-1}\mathcal{L}^{k+l-1}]_-\mathcal{P}_L\\
=\ & (km-nl)\partial^*_{m+n-1,k+l-1}\mathcal{P}_L.
\end{aligned}$$

Similarly the same results on $\mathcal{P}_R$ and $\mathcal{L}$ are as follows

$$\begin{aligned}
[\partial_{t^*_{m,l}},\partial_{t^*_{n,k}}]\mathcal{P}_R &= ((km-nl)(M_L-M_R)^{m+n-1}\mathcal{L}^{k+l-1})_+\mathcal{P}_R\\
&= (km-nl)\partial^*_{m+n-1,k+l-1}\mathcal{P}_R,\\
[\partial_{t^*_{m,l}},\partial_{t^*_{n,k}}]\mathcal{L} &= \partial_{t^*_{m,l}}(\partial_{t^*_{n,k}}\mathcal{L})-\partial_{t^*_{n,k}}(\partial_{t^*_{m,l}}\mathcal{L})\\
&= [((nl-km)(M_L-M_R)^{m+n-1}\mathcal{L}^{k+l-1})_-,\mathcal{L}]\\
&= (km-nl)\partial^*_{m+n-1,k+l-1}\mathcal{L}.
\end{aligned}$$

□

Denote $d_{m,l}=\partial_{t^*_{m+1,l+1}}$, and let $\mathcal{B}$ be the span of all $d_{m,l}$, $m,l\geq -1$. Then by (5.65), $\mathcal{B}$ is a Lie algebra with relations

$$[d_{m,l},d_{n,k}]=((m+1)(k+1)-(l+1)(n+1))d_{m+n,l+k},\quad \text{for}\ \ m,n,l,k\geq -1. \tag{5.66}$$

Thus $\mathcal{B}$ is in fact a Block type Lie algebra [24–31] (or more precisely the upper infinite part of a Block type Lie algebra [24]). We see that $\mathcal{B}$ is generated by the set

$$B=\{d_{-1,0},d_{0,-1},d_{0,0},d_{1,0},d_{0,1}\}=\{\partial^*_{0,1}=\partial_{0,0}=\partial_{1,0},\partial^*_{1,0},\partial^*_{1,1},\partial^*_{2,1},\partial^*_{1,2}\}. \tag{5.67}$$

It is easily to see that $\{d_{m,0},m\geq -1\}$ and $\{d_{0,m},m\geq -1\}$ both span half of the centerless Virasoro algebra or Witt algebra. It is a challenging work to get some explicit representations of general Block type Lie algebras. Actually, as we shall show in later sections it is highly nontrivial to do this even for $\mathcal{B}$.

To get the intuitive understanding of the properties of the additional symmetries of the BTH, we would like to give the first few flows of additional symmetry in the following. These examples

show that additional symmetry flows indeed explicitly depend on the coordinate variables $t_{\gamma,n}$ and x. For the $t^*_{1,0}$ flow, (5.57) implies

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial t^*_{1,0}} &= [-(M_L - M_R)_-, \mathcal{L}] \\ &= 1 + \sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n-1}\mathcal{L} + \sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n-1}\mathcal{L},\end{aligned}$$

which further leads to

$$\begin{cases}\dfrac{\partial u_i}{\partial t^*_{1,0}} &= -\sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n-1}u_i - \sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n-1}u_i, \quad i \neq 0, \\ \dfrac{\partial u_0}{\partial t^*_{1,0}} &= 1 - \sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n-1}u_0 - \sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n-1}u_0.\end{cases}$$

Similarly, after some computations, we get the $t^*_{1,1}$ flow

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial t^*_{1,1}} &= \sum_{i\geq 1}^{N+M}\frac{i}{N}u_{N-i}\Lambda^{N-i} - \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n}\mathcal{L} \\ &\quad -\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n}\mathcal{L},\end{aligned}$$

which leads to

$$\frac{\partial u_i}{\partial t^*_{1,1}} = \frac{N-i}{N}u_i - \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n}u_i - \sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n}u_i,$$

for $-M \leq i \leq N-1$.

Other equations of additional symmetry can be got in similar ways which will not be mentioned here.

6. Block type actions on functions P_L and P_R

In this section, we shall give some specific additional flows acting on wave operators $\mathcal{P}_L$, $\mathcal{P}_R$ and then on functions P_L, P_R which denote the symbol of $\mathcal{P}_L$and $\mathcal{P}_R$ respectively (see (4.45) and (4.46)).

As we all know, W-constraint which means additional flows of wave function become vanishing is a very useful tool to bring action of additional flow on wave functions into action on tau function. But for the BTH, it is not easy to get the action on tau function. Even Virasoro constraint is still difficult to get which will be shown in the last section of this paper. So we choose another kind of constraints which are weaker than the W-constraints to give representations of the BTH. It is named weak W-constraints which will be shown later. By these specific

actions on functions P_L, P_R, we will present two representations of the Block type Lie algebra $\mathcal{B}$ under this kind of so-called weak W-constraints.

By definition, then

$$\frac{\partial \mathcal{P}_L}{\partial t^*_{0,1}} = -\mathcal{L}_- \mathcal{P}_L = \partial_{0,0}\mathcal{P}_L, \tag{6.1}$$

which leads to

$$\begin{aligned}\frac{\partial P_L}{\partial_{t^*_{0,1}}} &= A^*_{L0,1}P_L, \quad \text{where}\\ A^*_{L0,1} &= \partial_{0,0}.\end{aligned} \tag{6.2}$$

Considering (5.55), the $t^*_{1,0}$ flow can be written as following

$$\begin{aligned}\frac{\partial \mathcal{P}_L}{\partial t^*_{1,0}} &= -\left(M_L - M_R\right)_- \mathcal{P}_L\\ &= -\mathcal{P}_L\frac{x}{N\epsilon}\Lambda^{-N} - \sum_{\alpha=2}^{N} t_{\alpha,0}(1-\frac{\alpha-1}{N})\mathcal{P}_L\Lambda^{1-\alpha} + \sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n-1}\mathcal{P}_L\\ &\quad + \sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n-1}\mathcal{P}_L.\end{aligned}$$

Because $[\mathcal{P}_L, \frac{x}{\epsilon}]z^{\frac{x}{\epsilon}} = z(\partial_z P_L)z^{\frac{x}{\epsilon}}$,

$$\begin{aligned}\frac{\partial P_L}{\partial t^*_{1,0}}z^{\frac{x}{\epsilon}} &= \frac{\partial \mathcal{P}_L}{\partial t^*_{1,0}}z^{\frac{x}{\epsilon}}\\ &= \Big[-(\frac{z^{1-N}}{N}\partial_z P_L) - \frac{x}{N\epsilon}z^{-N}P_L - \sum_{\alpha=2}^{N} t_{\alpha,0}(1-\frac{\alpha-1}{N})z^{1-\alpha}P_L\\ &\quad + \sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}(\partial_{\alpha,n-1}P_L) + \sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}(\partial_{\beta,n-1}P_L)\Big]z^{\frac{x}{\epsilon}}.\end{aligned}$$

Then the $t^*_{1,0}$ flow of function P_L is given as follows

$$\begin{aligned}\frac{\partial P_L}{\partial t^*_{1,0}} &= A^*_{L1,0}P_L, \quad \text{where}\\ A^*_{L1,0} &= -\frac{z^{1-N}}{N}\partial_z - \frac{x}{N\epsilon}z^{-N} - \sum_{\alpha=2}^{N} t_{\alpha,0}(1-\frac{\alpha-1}{N})z^{1-\alpha} + \sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n-1}\\ &\quad + \sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n-1}.\end{aligned} \tag{6.3}$$

By the same calculation, we can get the $t^*_{1,1}$ flow of wave operator $\mathcal{P}_L$ and function P_L as follows

$$\begin{aligned}
\frac{\partial \mathcal{P}_L}{\partial t^*_{1,1}} &= -\mathcal{P}_L \frac{x}{N\epsilon} + \frac{x}{N\epsilon}\mathcal{P}_L - \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n}\mathcal{P}_L \\
&\quad -\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n}\mathcal{P}_L, \\
\frac{\partial P_L}{\partial t^*_{1,1}} &= A^*_{L1,1}P_L, \quad \text{where} \\
A^*_{L1,1} &= -\frac{z}{N}\partial_z + \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n} + \sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n}. \quad (6.4)
\end{aligned}$$

Similarly, the $t^*_{1,2}$ flow of wave operator $\mathcal{P}_L$ and function P_L can be got as follows

$$\begin{aligned}
\frac{\partial \mathcal{P}_L}{\partial t^*_{1,2}} &= -\mathcal{P}_L \frac{x}{N\epsilon}\Lambda^N + \sum_{i+j\leq N}\omega_i\Lambda^{-i}\frac{x}{N\epsilon}\Lambda^N\Lambda^{-j}\omega'_j\mathcal{P}_L + \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n+1}\mathcal{P}_L \\
&\quad - \sum_{i+j<M}\tilde{\omega}_i\Lambda^i\frac{x}{M\epsilon}\Lambda^{-M}\Lambda^j\tilde{\omega}'_j\mathcal{P}_L + \sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n+1}\mathcal{P}_L, \\
\frac{\partial P_L}{\partial t^*_{1,2}} &= A^*_{L1,2}P_L, \quad \text{where} \\
A^*_{L1,2} &= -\frac{z^{N+1}}{N}\partial_z + (\frac{x}{N\epsilon}+\frac{x}{M\epsilon})\partial_{t_{1,0}} - \frac{1}{N}\sum_{i+j\leq N} i\omega_i z^{N-i-j}\omega'_j(x+(N-i-j)\epsilon)\Lambda^{N-i-j} \\
&\quad + \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n+1} + \sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n+1} \\
&\quad -\frac{1}{M}\sum_{i+j<M} i\tilde{\omega}_i z^{i+j-M}\tilde{\omega}'_j(x+(i+j-M)\epsilon)\Lambda^{i+j-M}. \quad (6.5)
\end{aligned}$$

The general Virosora flows are

$$\begin{aligned}
\frac{\partial P_L}{\partial t^*_{1,m+1}} &= A^*_{L1,m+1}P_L, \quad m=0,1,..., \quad \text{where} \\
A^*_{L1,m+1} &= -\frac{z^{mN+1}}{N}\partial_z + (\frac{x}{N\epsilon}+\frac{x}{M\epsilon})\partial_{t_{1,m-1}} \\
&\quad -\frac{1}{N}\sum_{i+j\leq mN} i\omega_i z^{mN-i-j}\omega'_j(x+(mN-i-j)\epsilon)\Lambda^{mN-i-j}
\end{aligned}$$

$$+\sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n+m}+\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n+m}$$
$$-\frac{1}{M}\sum_{i+j<mM}i\tilde{\omega}_i z^{i+j-mM}\tilde{\omega}'_j(x+(i+j-mM)\epsilon)\Lambda^{i+j-mM}.$$

Now we will consider the *weak W-constraints*[1]. For example, condition

$$\partial^*_{k,s}A^*_{L1,2}=0, \tag{6.7}$$

will lead to the following constraint

$$\partial^*_{k,s}\left(\sum_{i+j=l\leq N}i\omega_i\omega'_j(x+(N-l)\epsilon)\right)=0 \tag{6.8}$$

$$\partial^*_{k,s}\left(\sum_{i+j=s<M}i\tilde{\omega}_i\tilde{\omega}'_j(x+(s-M)\epsilon)\right)=0, \tag{6.9}$$

where $k,s,l,s\geq 0$.

Considering the weak W-constraints (6.6), these operators $\{A^*_{L1,m},m\geq 0\}$ can be regarded as a representation of the Virasoro algebra acting on P_L function space, i.e.,

$$[A^*_{L1,m},A^*_{L1,n}]P_L=(m-n)A^*_{L1,m+n-1}P_L. \tag{6.10}$$

After a tedious calculation, we can get the $t^*_{2,1}$ flow on function P_L

$$\frac{\partial P_L}{\partial t^*_{2,1}}=A^*_{L2,1}P_L, \tag{6.11}$$

where $A^*_{L2,1}$ has very complicated form which will be presented in (8.1) in the appendix.

Till now, we have given all generating elements (5.67) of the Block type Lie algebra $\mathcal{B}$ acting on function P_L, i.e. $\{A^*_{L0,1},A^*_{L1,0},A^*_{L1,1},A^*_{L2,1},A^*_{L1,2}\}$ in (6.2)–(6.11). Thus we have in fact given a representation of $\mathcal{B}$ on P_L function space.

Similarly corresponding flows of P_R can be got as follows. Firstly, by definition,

$$\frac{\partial\mathcal{P}_R}{\partial t^*_{0,1}}=\mathcal{L}_+\mathcal{P}_R=\partial_{1,0}\mathcal{P}_R,$$

[1]The weak W-constraints mean that the coefficients $\{\omega_i,\omega'_i,\tilde{\omega}_i,\tilde{\omega}'_i,i\geq 0\}$ $(\omega_0=\omega'_0=1)$ satisfy the vanishing property of derivatives of all operators $A^*_{Lm,n}$ and $A^*_{Rm,n}$(will be introduced later) with respect to additional time variables, i.e.

$$\partial^*_{k,s}A^*_{Lm,n}=\partial^*_{k,s}A^*_{Rm,n}=0, \tag{6.6}$$

for all $k,s,m,n\geq 0$. W-constraints can imply weak W-constraints. Therefore condition under weak W-constraints is weaker than W-constraints.

which leads to

$$\begin{aligned}\frac{\partial P_R}{\partial_{t^*_{0,1}}} &= A^*_{R0,1}P_R, \quad \text{where}\\ A^*_{R0,1} &= \partial_{1,0}.\end{aligned} \tag{6.12}$$

Secondly, by the following calculation

$$\begin{aligned}\frac{\partial \mathcal{P}_R}{\partial t^*_{1,0}} &= (M_L - M_R)_+ \mathcal{P}_R\\ &= \mathcal{P}_R \frac{x}{M\epsilon}\Lambda^M + \sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n-1}\mathcal{P}_R + t_{1,0}\mathcal{P}_R\\ &\quad + \sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n-1}\mathcal{P}_R + \sum_{\beta=-M+1}^{0}(1+\frac{\beta}{M})t_{\beta,0}\mathcal{P}_R\Lambda^{-\beta},\end{aligned}$$

and considering $[\mathcal{P}_R, \frac{x}{\epsilon}]z^{\frac{x}{\epsilon}} = z(\partial_z P_R)z^{\frac{x}{\epsilon}}$, we can get

$$\begin{aligned}\frac{\partial P_R}{\partial t^*_{1,0}} &= A^*_{R1,0}P_R, \quad \text{where}\\ A^*_{R1,0} &= \frac{z^{M+1}}{M}\partial_z + \frac{x}{M\epsilon}z^M + \sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n-1} + t_{1,0}\\ &\quad + \sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n-1} + \sum_{\beta=-M+1}^{0}(1+\frac{\beta}{M})t_{\beta,0}z^{-\beta}.\end{aligned} \tag{6.13}$$

In the same way,

$$\begin{aligned}\frac{\partial P_R}{\partial t^*_{1,1}} &= A^*_{R1,1}P_R, \quad \text{where}\\ A^*_{R1,1} &= \frac{z}{M}\partial_z + \frac{x}{M\epsilon} + \frac{x}{N\epsilon} + \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n}\\ &\quad + \sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n}.\end{aligned} \tag{6.14}$$

Similarly, the $t^*_{1,2}$ flows of P_R can be derived as follows

$$\begin{aligned}\frac{\partial P_R}{\partial t^*_{1,2}} &= A^*_{R1,2}P_R, \quad \text{where}\\ A^*_{R1,2} &= (\frac{x}{N\epsilon} + \frac{x}{M\epsilon})\partial_{t_{1,0}} - \frac{1}{N}\sum_{i+j\leq N} i\omega_i z^{N-i-j}\omega'_j(x+(N-i-j)\epsilon)\Lambda^{N-i-j}\end{aligned}$$

$$+\sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n+1}+\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n+1}$$
$$+\frac{1}{M}\sum_{i+j\geq M}i\tilde{\omega}_i z^{i+j-M}\tilde{\omega}'_j(x+(i+j-M)\epsilon)\Lambda^{i+j-M}.$$

Similarly, after a tedious calculation, we obtain the $t^*_{2,1}$ flow function P_R as follows

$$\frac{\partial P_R}{\partial t^*_{2,1}}=A^*_{R2,1}P_R, \tag{6.15}$$

where $A^*_{R2,1}$ has complicated form which will also be presented presented in (8.2) in the appendix.

Denote $A^*_{Lm+1,l+1}$ and $A^*_{Rm+1,l+1}$ by $D^*_{Lm,l}$ and $D^*_{Rm,l}$ respectively. Considering the weak W-constraints of functions $P_L(t,z)$ and $P_R(t,z)$, there exist two anti-isomorphisms between $\{d^*_{m,l}\}$ and $\{D^*_{Lm,l}\}$, $\{d^*_{m,l}\}$ and $\{D^*_{Rm,l}\}$ respectively as follows

$$\begin{array}{rrcl} P_L: & d^*_{m,l} & \mapsto & D^*_{Lm,l} \\ & \left[d^*_{m,l},d^*_{n,k}\right] & \mapsto & \left[D^*_{Ln,k},D^*_{Lm,l}\right], \\ P_R: & d^*_{m,l} & \mapsto & D^*_{Rm,l} \\ & \left[d^*_{m,l},d^*_{n,k}\right] & \mapsto & \left[D^*_{Rn,k},D^*_{Rm,l}\right]. \end{array}$$

The anti-isomorphisms under Lie bracket are

$$\left[d^*_{m,l},d^*_{n,k}\right]P_L=\left[D^*_{Ln,k},D^*_{Lm,l}\right]P_L,\quad \text{and}\quad \left[d^*_{m,l},d^*_{n,k}\right]P_R=\left[D^*_{Rn,k},D^*_{Rm,l}\right]P_R.$$

We can get other additional flows on wave operators and functions by very tedious calculations using recursion relation, which will not be mentioned here. It would be great to get the actions of additional flows on τ function which can lead to the Virasoro constraints. Although it is not easy to bring action onto tau functions under W-constraints, we can also get the following important theorem.

Theorem 6.1. (1) *Equations* (6.2)–(6.11) *give a representation of* $\mathcal{B}$ *on* P_L *function space.*
(2) *Equations* (6.12)–(6.15) *give a representation of* $\mathcal{B}$ *on* P_R *function space.*

It should be remarked that the two representations in Theorem 6.1 are both under the weak W-constraints. And also we would like to emphasize that it is quite nontrivial to give representations of Block Lie algebra in Theorem 6.1 from the point of representation of infinite dimensional Lie algebra. It is difficult to give one representation of this Block Lie algebra but luckily we get it from the first few additional flows of the BTH.

7. Conclusions and discussions

We define Orlov-Schulman's M_L, M_R operators of BTH in Section 4 and give the additional symmetries of the BTH in Section 5. Then we find that the additional symmetries form one

kind of Block type Lie algebra which has recently received much attention. In Section 6, we obtain some flows on functions P_L and P_R which lead to two representations of this Block type Lie algebra $\mathcal{B}$ which contains the half centerless Virasoro algebra (or Witt algebra) as a subalgebra.

The additional symmetries provide a very useful way to derive an explicit representations of the Virasoro algebra for KP hierarchy [13], BKP hierarchy [20, 21] and 2-D Toda hierarchy [15] by its action on the space of τ function. This representation is more elegant and simple by comparing with its representation defined on the space of functions P_L, P_R.

Therefore we tried to get an explicit representation of the algebra $\mathcal{B}$ in a similar way through the actions of the additional symmetries on the τ function of the BTH. Due to some technical reasons, it does not seem easy to derive additional actions on τ function under the W-constraints, we have thus only found the first few operators of the algebra $\mathcal{B}$. There are two ways to get the Virasoro representations from the point of view of the integrable system. One way is to use ASvM formula [15] and the other one is to use similar forms [16]. Here we would like to use the second way to derive first two operators in algebra $\mathcal{B}$ from acting on τ function. Firstly, to this purpose, it is useful to rewrite its action on the functions P_L, P_R as a similar form [16]. Because the τ function of BTH is defined by P_L as follows [11]

$$P_L := \frac{\tau(x, t - [z^{-1}]^N; \epsilon)}{\tau(x, t; \epsilon)}, \tag{7.1}$$

where

$$\left[z^{-1}\right]^N_{\alpha,n} := \begin{cases} \frac{z^{-N(n+1-\frac{\alpha-1}{N})}}{N(n+1-\frac{\alpha-1}{N})}, & \alpha = N, N-1, \dots 1, \\ 0, & \alpha = 0, -1 \cdots -(M-1), \end{cases}$$

a technical calculation [16] can lead to the following identity in a similar form on the τ function

$$\begin{aligned}
\frac{\partial P_L}{\partial t^*_{1,0}} &= A^*_{L1,0} P_L \\
&= \Big[(\frac{x}{\epsilon} + \frac{N-1}{2})(\tilde{t}_{1,0} - t_{1,0}) \\
&\quad + \sum_{\alpha=2}^{N} \frac{(\alpha-1)(N-\alpha+1)}{2N} ((t_{\alpha,0} - \frac{z^{-(N-\alpha+1)}}{N-\alpha+1})(t_{N+2-\alpha,0} - \frac{z^{1-\alpha}}{\alpha-1}) - t_{\alpha,0} t_{N+2-\alpha,0}) \\
&\quad + \sum_{n>0} \sum_{\alpha=1}^{N} (n+1-\frac{\alpha-1}{N})(t_{\alpha,n} - \frac{z^{-N(n+1-\frac{\alpha-1}{N})}}{N(n+1-\frac{\alpha-1}{N})}) \frac{\partial_{\alpha,n-1}\tilde{\tau}}{\tilde{\tau}} \\
&\quad - \sum_{n>0} \sum_{\alpha=1}^{N} (n+1-\frac{\alpha-1}{N}) t_{\alpha,n} \frac{\partial_{\alpha,n-1}\tau}{\tau}
\end{aligned}$$

$$+\sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}(\frac{\partial_{\beta,n-1}\tilde{\tau}}{\tilde{\tau}}-\frac{\partial_{\beta,n-1}\tau}{\tau})\Big]\frac{\tilde{\tau}}{\tau},$$

where τ denotes $\tau(x,t,\epsilon)$ and $\tilde{\tau}$ denotes $\tau(x,t-[z^{-1}]^N,\epsilon)$.

After supposing

$$\frac{\partial P_L}{\partial t^*_{1,0}} = 0,$$

and taking the integral constant (in fact this integral constant can depend on $t_{\beta,n}$ variables mentioned before) to be zero, we can find

$$\begin{aligned}
\mathcal{L}_{L1,0}\tau &= 0, \quad \text{where} \\
\mathcal{L}_{L1,0} &= (\frac{x}{\epsilon}+\frac{N-1}{2})t_{1,0}+\sum_{\alpha=2}^{N}\frac{(\alpha-1)(N-\alpha+1)}{2N}t_{\alpha,0}t_{N+2-\alpha,0} \\
&\quad +\sum_{n>0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n-1}+\sum_{n>0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n-1}.
\end{aligned}$$

Similarly, by the following calculation

$$\begin{aligned}
\frac{\partial P_L}{\partial t^*_{1,1}} &= A^*_{L1,1}P_L \\
&= \Big[\sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})(t_{\alpha,n}-\frac{z^{-N(n+1-\frac{\alpha-1}{N})}}{N(n+1-\frac{\alpha-1}{N})})\frac{\partial_{\alpha,n}\tilde{\tau}}{\tilde{\tau}} \\
&\quad -\sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\frac{\partial_{\alpha,n}\tau}{\tau}+\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}(\frac{\partial_{\beta,n}\tilde{\tau}}{\tilde{\tau}}-\frac{\partial_{\beta,n}\tau}{\tau})\Big]\frac{\tilde{\tau}}{\tau},
\end{aligned}$$

the second Virasoro operator can be got as following

$$\mathcal{L}_{L1,1} = \sum_{n\geq 0}\sum_{\alpha=1}^{N}(n+1-\frac{\alpha-1}{N})t_{\alpha,n}\partial_{\alpha,n}+\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}(n+1+\frac{\beta}{M})t_{\beta,n}\partial_{\beta,n}.$$

Only some of the Virasoro elements can be explicitly constructed because for the higher additional flows, similar forms as in [16] are too complicated. For example, we did not have the similar form of the $A^*_{L1,2}$ as in (6.5). The above are about the actions on P_L. Similar situations occur for the actions on P_R, which will not be mentioned here. Our future work will contain ASvM formula for the BTH, which might further lead to some representations of that Block type Lie algebra.

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References

[1] M. Toda, Vibration of a chain with nonlinear interaction, J. Phys. Soc. Jpn., 22(1967), 193-198.
[2] M. Sato, Y. Sato, "Soliton equations as dyanmical systems on infinite dimensional Grassmann manifold", In: Nonlinear Partial Differential Equations in Applied Science: Proceedings of The U.S.-Japan Seminar (Tokyo, 1982), Lecture Notes in Num. Appl. Anal. 5, edited by H. Fujita, P. D Lax and G. Strang (North-Holland, Amsterdam, 1983) , pp. 259-271.
[3] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, "Transformation groups for soliton equations". In: Proceedings of RIMS Symposium on Non-linear Integrable Systems-Classical Theory and Quantum Theory (Kyoto,1981), edited by M. Jimbo and T. Miwa (World Scientifcic, Singapore, 1983), pp. 39-119.
[4] K. Ueno, K. Takasaki, Toda lattice hierarchy, In *"Group representations and systems of differential equations" (Tokyo, 1982)*, 1-95, Adv. Stud. Pure Math., 4, North-Holland, Amsterdam, 1984.
[5] C. Z. Li, Solutions of bigraded Toda hierarchy, Journal of Physics A: Mathematical and Theoretical, 44, 255201(2011).
[6] G. Carlet, B. Dubrovin, Y. Zhang, The Extended Toda Hierarchy, Mosc. Math. J.,4(2004), 313-332.
[7] B. Dubrovin, Y. Zhang, Virasoro symmetries of the extended Toda hierarchy, Comm. Math. Phys., 250(2004), 161-193.
[8] G. Carlet, The extended bigraded Toda hierarchy, J. Phys. A, 39(2006), 9411-9435, arXiv:math-ph/0604024.
[9] S. Aoyama, Y. Kodama, Topological Landau-Ginzburg theory with a rational potential and the dispersionless KP hierarchy, Comm. Math. Phys., 182(1996), 185-219.
[10] T. Milanov, H. H. Tseng, The spaces of Laurent polynomials, $\mathbb{P}^1$-orbifolds, and integrable hierarchies, Journal für die reine und angewandte Mathematik, 622(2008), 189-235. arXiv:math.AG/0607012.
[11] C. Z. Li, J. S. He, K. Wu, Y. Cheng, Tau function and Hirota bilinear equations for the extended bigraded Toda Hierarchy, J. Math. Phys. 51,043514(2010), arXiv: 0906.0624.
[12] G. Carlet, B. Dubrovin, L.P. Mertens, Infinite-dimensional Frobenius manifold for 2+1 integrable systems, arXiv: 0902.1245.
[13] A. Yu. Orlov, E. I. Schulman, Additional symmetries of integrable equations and conformal algebra reprensentaion, Lett. Math. Phys., 12(1986), 171-179.
[14] M. Adler, P. V. Moerbeke, A matrix integral Solution to two-dimensional W_p-Gravity, Commun. Math. Phys., 147(1992), 25-56.
[15] M. Adler, T. Shiota, P. van Moerbeke, A Lax representation for the vertex operator and the central extension, Commun. Math. Phys., 171(1995), 547-588.
[16] L. A. Dickey, Additional symmetries, Grassmannian, and string equation, Modern physic letter A, 8(1993), 1259-1272.
[17] L. A. Dickey, Additional symmetries, Grassmannian, and string equation II, Modern physic letter A, 8(1993), 1357-1377.
[18] L. A. Dickey, On additional symmetries of the KP hierarchy and Sato's Backlund transformation, Commun. Math. Phys., 167 (1995), 227-233.
[19] K. Takasaki, T. Takebe, Quasi-classical limit of Toda hierarchy and W-infinity symmetries, Lett. Math. Phys., 28(1993), 165-176.
[20] M. H. Tu, On the BKP Hierarchy: additional symmetries, Fay identity and Adler-Shiota-van Moerbeke formula, Lett. Math. Phys., 81(2007),93-105.
[21] H. F. Shen, M. H. Tu, On the string equation of the BKP hierarchy, International Journal of Modern Physics A, 24(2009), 4193-4208.

[22] J. S. He, K. L. Tian, A. Foerster, W. X. Ma, Additional symmetries and string equation of the CKP Hierarchy, Lett. Math. Phys., 81(2007), 119-134.
[23] S. Aoyama, Y. Kodama, A generalized Sato equation and the W_∞ algebra, Phys. Let. B, 278(1992), 56-62.
[24] R. Block, On torsion-free abelian groups and Lie algebras, Proc. Amer. Math. Soc., 9(1958), 613-620.
[25] D. Dokovic, K. Zhao, Derivations, isomorphisms and second cohomology of generalized Block algebras, Algebra Colloq., 3(1996), 245-272.
[26] J. M. Osborn, K. Zhao, Infinite-dimensional Lie algebras of generalized Block type, Proc. Amer. Math. Soc., 127(1999), 1641-1650.
[27] X. Xu, Generalizations of Block algebras, Manuscripta Math., 100(1999), 489-518.
[28] X. Xu, Quadratic conformal superalgebras, J. Algebra, 224(2000), 1-38.
[29] H. Zhang, K. Zhao, Representations of the Virasoro-like Lie algebra and its q-analog, Comm. Algebra, 24(1996), 4361-4372.
[30] L. Zhu, D. Meng, Structure of degenerate Block algebras, Algebra Colloq., 10(2003), 53-62.
[31] Y. Su, Quasifinite representations of a Lie algebra of Block type, J. Algebra, 276(2004), 117-128.

8. Appendix

After a tedious calculation, we can get the $t^*_{2,1}$ flow on wave function P_L,

$$\frac{\partial P_L}{\partial t^*_{2,1}} = A^*_{L2,1} P_L,$$

where,

$$\begin{aligned}
& A^*_{L2,1} \qquad (8.1)\\
= \; & -\sum_{i=0}^{\infty} \omega_i \frac{x-i\epsilon}{N\epsilon}\frac{x-i\epsilon-N\epsilon}{N\epsilon} z^{-(N+i)}\Lambda^{-(N+i)} \\
& -\sum_{n\geq 0}\sum_{\alpha=1}^{N}\sum_{i+j>N(n-\frac{\alpha-1}{N})} \omega_i \Big(2\frac{x}{N\epsilon}+n-\frac{\alpha-1+2i}{N}\Big) z^{N(n-\frac{\alpha-1}{N})-i-j} \\
& t_{\alpha,n}\omega'_j\Big(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon\Big)\Lambda^{N(n-\frac{\alpha-1}{N})-i-j} \\
& +\sum_{n,n'\geq 0}\sum_{\alpha,\alpha'=1}^{N}\Big(n+1-\frac{\alpha-1}{N}\Big)\Big(n'+1-\frac{\alpha'-1}{N}\Big) t_{\alpha,n}t_{\alpha',n'}\partial_{\alpha+\alpha'-1,n+n'} \\
& -\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}\sum_{i+j<M(n+\frac{\beta}{M})}\Big(n+1+\frac{\beta}{M}\Big)\tilde{\omega}_i\Big(2\frac{x}{M\epsilon}-n-\frac{\beta-2i}{M}\Big) z^{-M(n+\frac{\beta}{M})+i+j} t_{\beta,n} \\
& \tilde{\omega}'_j\Big(x+[i+j-M(n+\frac{\beta}{M})]\epsilon\Big)\Lambda^{-M(n+\frac{\beta}{M})+i+j} \\
& +\sum_{n,n'\geq 0}\sum_{\beta,\beta'=-M+1}^{0}\Big(n+1+\frac{\beta}{M}\Big)\Big(n'+1+\frac{\beta'}{M}\Big) t_{\beta,n}t_{\beta',n'}\partial_{\beta+\beta',n+n'}
\end{aligned}$$

$$-\sum_{k+l<i+j+N}\omega_i\frac{x-i\epsilon}{N\epsilon}\omega'_j(x-(N+i+j)\epsilon)\tilde{\omega}_k(x-(N+i+j)\epsilon)\frac{x-(N+i+j-k)\epsilon}{M\epsilon}$$
$$\tilde{\omega}'_l(x+(k+l-N-i-j)\epsilon)z^{k+l-N-i-j}\Lambda^{k+l-N-i-j}$$
$$-\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}\sum_{k+l<i+j+N+M(n+1+\frac{\beta}{M})}\omega_i\frac{x-i\epsilon}{N\epsilon}\omega'_j(x-(N+i+j)\epsilon)\tilde{\omega}_k(x-(N+i+j)\epsilon)$$
$$(n+1+\frac{\beta}{M})z^{k+l-N-i-j-M(n+1+\frac{\beta}{M})}t_{\beta,n}\tilde{\omega}'_l(x+[k+l-N-i-j-M(n+1+\frac{\beta}{M})]\epsilon)$$
$$\Lambda^{k+l-N-i-j-M(n+1+\frac{\beta}{M})}$$
$$-\sum_{n\geq 0}\sum_{\alpha=1}^{N}\sum_{k+l<i+j-N(n-\frac{\alpha-1}{N})}\omega_i(n+1-\frac{\alpha-1}{N})z^{k+l+N(n-\frac{\alpha-1}{N})-i-j}\omega'_j(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)$$
$$\tilde{\omega}_k(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)\frac{x+[k+N(n-\frac{\alpha-1}{N})-i-j]\epsilon}{M\epsilon}$$
$$\tilde{\omega}'_l(x+[k+l+N(n-\frac{\alpha-1}{N})-i-j]\epsilon)\Lambda^{k+l+N(n-\frac{\alpha-1}{N})-i-j}$$
$$-\sum_{n,m\geq 0}\sum_{\alpha=1}^{N}\sum_{\beta=-M+1}^{0}(n+1-\frac{\alpha-1}{N})(m+1+\frac{\beta}{M})\sum_{k+l<i+j-N(n-\frac{\alpha-1}{N})+M(m+1+\frac{\beta}{M})}$$
$$\omega_i\omega'_j(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)\tilde{\omega}_k(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)$$
$$z^{N(n-\frac{\alpha-1}{N})-i-j+k+l-M(m+1+\frac{\beta}{M})}t_{\alpha,n}t_{\beta,m}$$
$$\tilde{\omega}'_l(x+[N(n-\frac{\alpha-1}{N})-i-j+k+l-M(m+1+\frac{\beta}{M})]\epsilon)\Lambda^{N(n-\frac{\alpha-1}{N})-i-j+k+l-M(m+1+\frac{\beta}{M})}$$
$$-\sum_{i+j+M<k+l}\tilde{\omega}_i\frac{x+i\epsilon}{M\epsilon}\tilde{\omega}'_j(x+(i+j+M)\epsilon)\omega_k(x+(i+j+M)\epsilon)$$
$$\frac{x+(i+j+M-k)\epsilon}{N\epsilon}\omega'_l(x+(i+j+M-k-l)\epsilon)z^{i+j+M-k-l}\Lambda^{i+j+M-k-l}$$
$$-\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}\sum_{i+j<k+l+M(n+\frac{\beta}{M})}\tilde{\omega}_i(n+1+\frac{\beta}{M})z^{i+j-M(n+\frac{\beta}{M})-k-l}t_{\beta,n}\tilde{\omega}'_j(x+[i+j-M(n+\frac{\beta}{M})]\epsilon)$$
$$\omega_k(x+[i+j-M(n+\frac{\beta}{M})]\epsilon)\frac{x+[i+j-k-M(n+\frac{\beta}{M})]\epsilon}{N\epsilon}$$
$$\omega'_l(x+[i+j-k-l-M(n+\frac{\beta}{M})]\epsilon)\Lambda^{i+j-k-l-M(n+\frac{\beta}{M})}$$
$$-\sum_{n\geq 0}\sum_{\alpha=1}^{N}\sum_{i+j+M<k+l-N(n+1+\frac{\alpha-1}{N})}\tilde{\omega}_i\frac{x+i\epsilon}{M\epsilon}\tilde{\omega}'_j(x+(i+j+M)\epsilon)\omega_k(x+(i+j+M)\epsilon)$$
$$(n+1-\frac{\alpha-1}{N})z^{N(n+1-\frac{\alpha-1}{N})-k-l+i+j+M}\omega'_l(x+(i+j+M+N(n+1-\frac{\alpha-1}{N})-k-l)\epsilon)$$

$$\Lambda^{i+j+M+N(n+1-\frac{\alpha-1}{N})-k-l} - \sum_{n,m\geq 0}\sum_{\alpha=1}^{N}\sum_{\beta=-M+1}^{0}(n+1-\frac{\alpha-1}{N})(m+1+\frac{\beta}{M})t_{\alpha,n}t_{\beta,m}$$
$$\sum_{k+l>i+j+N(n+1-\frac{\alpha-1}{N})-M(m+\frac{\beta}{M})}\tilde{\omega}_i\tilde{\omega}'_j(x+[i+j-M(m+\frac{\beta}{M})]\epsilon)\omega_k(x+[i+j-M(m+\frac{\beta}{M})]\epsilon)$$
$$z^{i+j-M(m+\frac{\beta}{M})+N(n+1-\frac{\alpha-1}{N})-k-l}\omega'_l(x+[i+j-M(m+\frac{\beta}{M})+N(n+1-\frac{\alpha-1}{N})-k-l]\epsilon)$$
$$\Lambda^{i+j-M(m+\frac{\beta}{M})+N(n+1-\frac{\alpha-1}{N})-k-l}.$$

Similarly, after a tedious calculation, we can get the $t^*_{2,1}$ flow function P_R as follows

$$\frac{\partial P_R}{\partial t^*_{2,1}} = A^*_{R2,1}P_R,$$

where,

$$\begin{aligned}
&A^*_{R2,1} \qquad\qquad (8.2)\\
=\ &\sum_{i=0}^{\infty}\tilde{\omega}_i\frac{x+i\epsilon}{M\epsilon}\frac{x+i\epsilon+M\epsilon}{M\epsilon}z^{M+i}\Lambda^{M+i}\\
&+\sum_{n\geq 0}\sum_{\alpha=1}^{N}\sum_{i+j\leq N(n-\frac{\alpha-1}{N})}(n+1-\frac{\alpha-1}{N})\omega_i(2\frac{x}{N\epsilon}+n-\frac{\alpha-1+2i}{N})z^{N(n-\frac{\alpha-1}{N})-i-j}\\
&t_{\alpha,n}\omega'_j(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)\Lambda^{N(n-\frac{\alpha-1}{N})-i-j}\\
&+\sum_{n,n'\geq 0}\sum_{\alpha,\alpha'=1}^{N}(n+1-\frac{\alpha-1}{N})(n'+1-\frac{\alpha'-1}{N})t_{\alpha,n}t_{\alpha',n'}\partial_{\alpha+\alpha'-1,n+n'}\\
&+\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}\sum_{i+j\geq -M(n+\frac{\beta}{M})}(n+1+\frac{\beta}{M})\tilde{\omega}_i(2\frac{x}{M\epsilon}-n-\frac{\beta-2i}{M})z^{-M(n+\frac{\beta}{M})+i+j}t_{\beta,n}\\
&\tilde{\omega}'_j(x+[i+j-M(n+\frac{\beta}{M})]\epsilon)\Lambda^{-M(n+\frac{\beta}{M})+i+j}\\
&+\sum_{n,n'\geq 0}\sum_{\beta,\beta'=-M+1}^{0}(n+1+\frac{\beta}{M})(n'+1+\frac{\beta'}{M})t_{\beta,n}t_{\beta',n'}\partial_{\beta+\beta',n+n'}\\
&+\sum_{k+l\geq i+j+N}\omega_i\frac{x-i\epsilon}{N\epsilon}\omega'_j(x-(N+i+j)\epsilon)\tilde{\omega}_k(x-(N+i+j)\epsilon)\frac{x-(N+i+j-k)\epsilon}{M\epsilon}\\
&\omega'_l(x+(k+l-N-i-j)\epsilon)z^{k+l-N-i-j}\Lambda^{k+l-N-i-j}\\
&+\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}\sum_{k+l\geq i+j+N+M(n+1+\frac{\beta}{M})}\omega_i\frac{x-i\epsilon}{N\epsilon}\omega'_j(x-(N+i+j)\epsilon)\tilde{\omega}_k(x-(N+i+j)\epsilon)
\end{aligned}$$

$$z^{k+l-N-i-j-M(n+1+\frac{\beta}{M})}t_{\beta,n}\tilde{\omega}'_l(x+[k+l-N-i-j-M(n+1+\frac{\beta}{M})]\epsilon)$$
$$\Lambda^{k+l-N-i-j-M(n+1+\frac{\beta}{M})}$$
$$+\sum_{n\geq 0}\sum_{\alpha=1}^{N}\sum_{k+l\geq i+j-N(n-\frac{\alpha-1}{N})}\omega_i(n+1-\frac{\alpha-1}{N})z^{k+l+N(n-\frac{\alpha-1}{N})-i-j}\omega'_j(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)$$
$$\tilde{\omega}_k(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)\frac{x+[k+N(n-\frac{\alpha-1}{N})-i-j]\epsilon}{M\epsilon}$$
$$\tilde{\omega}'_l(x+[k+l+N(n-\frac{\alpha-1}{N})-i-j]\epsilon)\Lambda^{k+l+N(n-\frac{\alpha-1}{N})-i-j}$$
$$+\sum_{n,m\geq 0}\sum_{\alpha=1}^{N}\sum_{\beta=-M+1}^{0}(n+1-\frac{\alpha-1}{N})(m+1+\frac{\beta}{M})\sum_{k+l\geq i+j-N(n-\frac{\alpha-1}{N})+M(m+1+\frac{\beta}{M})}$$
$$\omega_i\omega'_j(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)\tilde{\omega}_k(x+[N(n-\frac{\alpha-1}{N})-i-j]\epsilon)$$
$$z^{N(n-\frac{\alpha-1}{N})-i-j+k+l-M(m+1+\frac{\beta}{M})}t_{\alpha,n}t_{\beta,m}$$
$$\tilde{\omega}'_l(x+[N(n-\frac{\alpha-1}{N})-i-j+k+l-M(m+1+\frac{\beta}{M})]\epsilon)\Lambda^{N(n-\frac{\alpha-1}{N})-i-j+k+l-M(m+1+\frac{\beta}{M})}$$
$$+\sum_{i+j+M\geq k+l}\tilde{\omega}_i\frac{x+i\epsilon}{M\epsilon}\tilde{\omega}'_j(x+(i+j+M)\epsilon)\omega_k(x+(i+j+M)\epsilon)$$
$$\frac{x+(i+j+M-k)\epsilon}{N\epsilon}\omega'_l(x+(i+j+M-k-l)\epsilon)z^{i+j+M-k-l}\Lambda^{i+j+M-k-l}$$
$$+\sum_{n\geq 0}\sum_{\beta=-M+1}^{0}\sum_{i+j\geq k+l+M(n+\frac{\beta}{M})}\tilde{\omega}_i(n+1+\frac{\beta}{M})$$
$$z^{i+j-M(n+\frac{\beta}{M})-k-l}t_{\beta,n}\tilde{\omega}'_j(x+[i+j-M(n+\frac{\beta}{M})]\epsilon)$$
$$\omega_k(x+[i+j-M(n+\frac{\beta}{M})]\epsilon)\frac{x+[i+j-k-M(n+\frac{\beta}{M})]\epsilon}{N\epsilon}$$
$$\omega'_l(x+[i+j-k-l-M(n+\frac{\beta}{M})]\epsilon)\Lambda^{i+j-k-l-M(n+\frac{\beta}{M})}$$
$$+\sum_{n\geq 0}\sum_{\alpha=1}^{N}\sum_{i+j+M\geq k+l-N(n+1-\frac{\alpha-1}{N})}\tilde{\omega}_i\frac{x+i\epsilon}{M\epsilon}\tilde{\omega}'_j(x+(i+j+M)\epsilon)\omega_k(x+(i+j+M)\epsilon)$$
$$(n+1-\frac{\alpha-1}{N})z^{N(n+1-\frac{\alpha-1}{N})-k-l+i+j+M}\omega'_l(x+(i+j+M+N(n+1-\frac{\alpha-1}{N})-k-l)\epsilon)$$
$$\Lambda^{i+j+M+N(n+1-\frac{\alpha-1}{N})-k-l}+\sum_{n,m\geq 0}\sum_{\alpha=1}^{N}\sum_{\beta=-M+1}^{0}(n+1-\frac{\alpha-1}{N})(m+1+\frac{\beta}{M})t_{\alpha,n}t_{\beta,m}$$

$$\sum_{k+l\leq i+j+N(n+1-\frac{\alpha-1}{N})-M(m+\frac{\beta}{M})} \tilde{\omega}_i\tilde{\omega}'_j(x+[i+j-M(m+\frac{\beta}{M})]\epsilon)\omega_k(x+[i+j-M(m+\frac{\beta}{M})]\epsilon)$$
$$z^{i+j-M(m+\frac{\beta}{M})+N(n+1-\frac{\alpha-1}{N})-k-l}\omega'_l(x+[i+j-M(m+\frac{\beta}{M})+N(n+1-\frac{\alpha-1}{N})-k-l]\epsilon)$$
$$\Lambda^{i+j-M(m+\frac{\beta}{M})+N(n+1-\frac{\alpha-1}{N})-k-l}.$$